Pre-entry Test for MATH103/EMTH119: Solutions

The number of marks for each question is stated at the start of each question. Each answer in a question is worth one mark. The test is marked out of a total of 44. You need to get **at least** 33/44.

1. **[4 marks]**

   If $f(x) = e^x$ and $g(x) = 5x + 1$

   (a) $g(x+2) = 5(x+2)+1 = 5x+11$

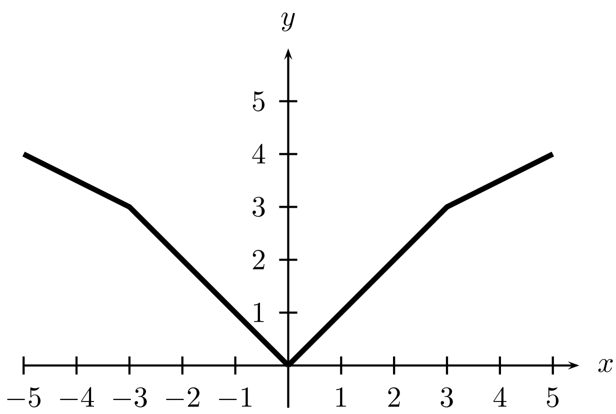
   (b) $g(x)+2 = (5x+1)+2 = 5x+3$

   (c) $f_\circ g(x) = f(5x+1) = e^{5x+1}$

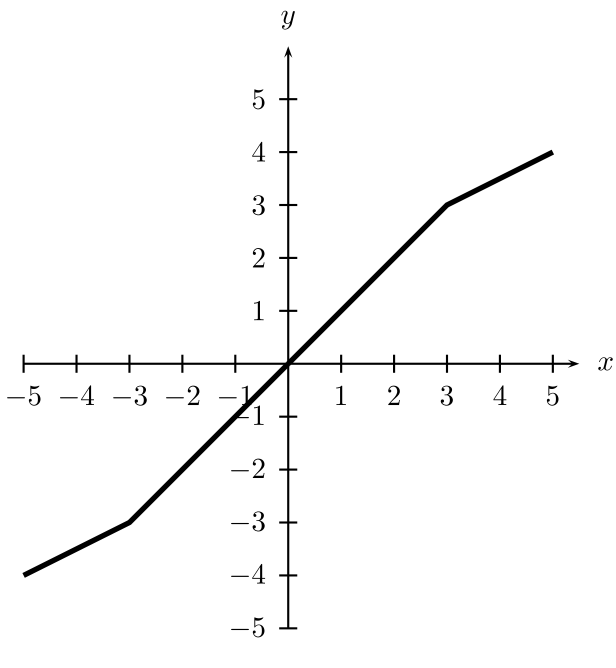
   (d) $g_\circ f(x) = g(e^x) = 5e^x + 1$

2. **[2 marks]**

   (a) The graph of f so that f is even is:



   (b) The graph of f so that f is odd is:



3. **[3 marks]**

(a) Since

$$f'(x) = \frac{1}{2\sqrt{x+4}} > 0$$

on $(-4, \infty)$, f is strictly increasing on this interval and will therefore be one-to-one and will have an inverse on $[-4, \infty)$.

(b) The inverse function is given by $f^{-1}(x) = x^2 - 4$.

(c) The domain of f^{-1} is $[0, \infty)$ and range of f^{-1} is $[-4, \infty)$.

4. **[4 marks]**

(a) This limit does not exist. We can write $\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$.

(b) $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$.

(c) This limit does not exist. We can write $\lim_{x \to \infty} \frac{x^3 - 2x + 1}{2x^2 - 3} = \infty$.

(d) $\lim_{x \to \infty} \frac{3 - x^2}{1 + 2x^2} = -\frac{1}{2}$.

5. **[4 marks]**

(a) $f'(x) = -2x \sin(x^2 - 4)$.

(b) $\frac{dy}{dx} = -\frac{6x^2}{(1 + x^3)^3}$.

(c) $g'(z) = -\frac{\sin z}{\cos z} = -\tan z$.

(d) $\frac{dy}{dx} = 3x^2 e^{x^3+1}$.

6. **[1 mark]**

$$\frac{dy}{dx} = \frac{1 - \cos(x+y)}{\cos(x+y)}$$

7. **[3 marks]**

(a) The slope of the tangent at a general point (a, b) on the ellipse is given by

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}.$$

(b) We want the points where $y = 0$. These are $(\pm\sqrt{3}, 0)$.

(c) The tangent lines are parallel at these points since both lines have the same slope, $\frac{dy}{dx} = 2$.

8. **[2 marks]**

(a) $\int x\,\sqrt{1+x^2}\,dx \;=\; \frac{1}{3}(1+x^2)^{\frac{3}{2}} \;+\; C$ (b) $\int \frac{u}{u^2+7}\,du \;=\; \frac{1}{2}\ln(u^2+7) \;+\; C$

9. **[2 marks]**

(a) $\int x^2\,\ln(x)\,dx \;=\; \frac{1}{3}x^3\,\ln(x) - \frac{1}{9}x^3 + C$ (b) $\int x\,\cos(x)\,dx \;=\; x\,\sin(x) + \cos(x) + C$

10. **[3 marks]**

(a) The corresponding linear system has a unique (one) solution because the number of variables (three) in the system matches the number of pivots in the row echelon form of the augmented matrix.

(b) The row operation $R_3 \leftarrow R_3 - 2R_2$ takes the augmented matrix to

$$\left[\begin{array}{ccc:c} 3 & -2 & 0 & 1 \\ 1 & 2 & -3 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right].$$

The last row corresponds to the equation $0z = 2$ which has no solution. The system is inconsistent.

(c) The corresponding linear system has an infinite number of solutions because there are more variables than equations in it. The most pivots we can get is three but there are four variables. At least one variable is "free" giving rise to a parameter in the solution. (Note: This is a homogeneous system so the "no solution" case is not possible.)

11. **[4 marks]**

(a) $A \;-\; 3B \;=\; \begin{bmatrix} 1 & -13 \\ -23 & 2 \end{bmatrix}$

(b) The product $A\,C$ is not possible to compute as the matrices do not conform for this multiplication.

(c) $B\,D \;=\; \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(d) $B\,A \;-\; (DC)^T \;=\; \begin{bmatrix} -6 & 14 \\ 5 & -23 \end{bmatrix}$

12. **[2 marks]**

(a) This matrix is not invertible.

(b) $B^{-1} \;=\; \begin{bmatrix} 0.5 & 0.5 & -0.5 \\ 0 & 0 & 1 \\ 0.5 & -0.5 & 0.5 \end{bmatrix}$

13. **[4 marks]**

(a) $\mathbf{u} \cdot \mathbf{v} \;=\; 10.$

(b) $\mathbf{u} \;\times\; \mathbf{v} \;=\; (4,\, -8,\, -10).$

(c) $||\mathbf{u}|| \;=\; \sqrt{20}$ and $||\mathbf{v}|| \;=\; \sqrt{14}.$

(d) The angle between $\mathbf{u}$ and $\mathbf{v}$ is 0.93 radians (to 2 decimal places).

14. **[3 marks]**

(a) A direction vector for this line is

$$\mathbf{d} \;=\; (1,\, -1,\, 2) \;-\; (3,\, -2,\, 1) \;=\; (-2,\, 1,\, 1)$$

so one possible vector parametric description is (and there are plenty of others!)

$$\mathbf{r} \;=\; (1,\, -1,\, 2) \;+\; s\,(-2,\, 1,\, 1)$$

(b) The line given by $x \;=\; 3+t$, $y \;=\; 4$ and $z \;=\; 2t$ can be written in the form

$$\mathbf{r} \;=\; (3,\, 4,\, 0) \;+\; t\,(1,\, 0,\, 2)$$

so a direction vector for this line is $\mathbf{d} \;=\; (1,\, 0,\, 2)$.

Since the lines are parallel, we can use $(1,\, 0,\, 2)$ for the direction vector for the line, say $\mathbf{w}$, that we are after. That is, a vector parametric equation for $\mathbf{w}$ is given by

$$\mathbf{w} \;=\; (1,\, 0,\, -1) \;+\; s\,(1,\, 0,\, 2)$$

(c) Setting $y = t$, where t is an arbitrary parameter, gives $x = 4 - 3t$.
A vector parametric form for this line is

$$\mathbf{r} \;=\; (4-3t,\, t) \;=\; (4,\, 0) \;+\; t\,(-3,\, 1)$$

15. **[3 marks]**

(a) A normal for the plane $3x - y + 5z = -3$ is $\mathbf{n} \;=\; (3,\, -1,\, 5)$.

(b) Two direction vectors for the plane are given by $\mathbf{d} \;=\; (1,\, -1,\, 0)$ and $\mathbf{e} \;=\; (1,\, 0,\, -1)$, so a parametric form for the plane is

$$\mathbf{r} \;=\; (1,\, 0,\, 0) \;+\; s(1,\, -1,\, 0) \;+\; t(1,\, 0,\, -1)$$

(c) A normal for the plane in (b) is given by

$$\mathbf{n} \;=\; (-1,\, 1,\, 0) \;\times\; (-1,\, 0,\, 1) \;=\; (1,\, 1,\, 1)$$

So the vector point-normal form of the plane is

$$(\mathbf{r} - (1,\, 0,\, 0)) \;\cdot\; (1,\, 1,\, 1) \;=\; 0$$

Setting $\mathbf{r} \;=\; (x,\, y,\, z)$ gives the scalar point-normal form

$$x \;+\; y \;+\; z \;=\; 1\,.$$